

# RIEFFEL'S DEFORMATION QUANTIZATION AND ISOSPECTRAL DEFORMATIONS

ANDRZEJ SITARZ

ABSTRACT. We demonstrate the relation between the isospectral deformation and Rieffel's deformation quantization by the action of  $\mathbb{R}^d$ .

## 1. INTRODUCTION

The isospectral deformation has been recently introduced by Connes and Landi [3], with the examples of the noncommutative 3 and 4 spheres, providing new examples of noncommutative spectral triples [2]. However, the construction of similar examples, though in different context, can be found in earlier works by Kulish and Mudrov [5] as well as, more recently in Paschke Ph.D. thesis [6].

The properties of isospectral deformations has been recently a subject of much interest [4]. In particular, the symmetries of isospectral geometries (seen as Hopf algebras acting on the deformed algebras) appear to be a twist by a Cartan subalgebra of the universal enveloping algebra of Lie algebras, which were symmetries of the undeformed algebras[12], and the constructed spectral triples are symmetric in the sense of [7].

The analysis of symmetries and the generalized construction of the deformation, as well as the similarity of the construction with the well-known noncommutative torus has led to the connections between the *isospectral deformations* and Rieffel's [8] deformation quantization by the action of  $\mathbb{R}^n$ . In fact, both the general construction as well as the corresponding symmetries were already defined in a general setup by Rieffel (see also [9, 10, 11]).

In this Letter we show that the isospectral deformations as defined in [3] are the special case of the Rieffel's construction [8].

## 2. THE ISOSPECTRAL DEFORMATIONS.

In the original article [3] one starts with the algebra  $\mathcal{A}$  of smooth functions on a manifold, with an isometry group of rank  $r \geq 2$ . This is

---

<sup>†</sup> Supported by Marie Curie Fellowship.

equivalent to the statement that the torus  $T^2$  is the isometry subgroup of the algebra. Now, taking the elements  $t$ , which are of  $C^\infty$  class with respect to the action of the torus group:  $t \mapsto \alpha_s(t)$ , one can obtain their decomposition as a norm convergent sum of homogeneous elements of a given bidegree. Then, for any homogeneous element one may define a deformed product (by a left or right twist). By linearity this extends to the linear combinations of homogeneous elements which are dense in the algebra we have started with.

To make correspondence with the Rieffel's deformation quantization we shall use the dual picture of isometries as used in [12].

*Remark 1.* Let  $\mathcal{A}$  be an algebra and the torus  $T^2$  be a subgroup of automorphisms of  $\mathcal{A}$ . Then for every operator  $T$ , which is of class  $C^\infty$  relative to the isometry  $\alpha_s$ , the additive group  $\mathbb{R}^2$  acts as follow:

$$(1) \quad [x_1, x_2] \triangleright t = \alpha_{e^{2\pi i x_1}, e^{2\pi i x_2}}(t),$$

Using the generators of torus symmetries  $p_1, p_2$ , for instance:

$$p_1 \triangleright T = \frac{1}{2\pi i} \left( \frac{d}{dx_1} ([x_1, x_2] \triangleright t) \right)_{|x=0},$$

the relation (1) could be rewritten as:

$$[x_1, x_2] \triangleright t = e^{2\pi i(x_1 p_1 + x_2 p_2)} \triangleright t.$$

*Remark 2.* With respect to the action of  $p_1, p_2$  the operators, which are homogeneous of degree  $(n_1, n_2)$ , behave like:

$$(2) \quad p_1 \triangleright t = n_1 t, \quad p_2 \triangleright t = n_2 t.$$

*Remark 3.* The product in the algebra  $\mathcal{A}$  can be deformed, first on elements of given degree, and then extending the deformation by linearity:

$$(3) \quad a * b = ab\lambda^{n_1^a n_2^b},$$

where  $\lambda$  is a complex number such that  $|\lambda| = 1$ . This gives the right twist of [3]. For future reference we shall denote the deformed algebra by  $\mathcal{A}_\lambda$ .

This deformation, can be extended, in the case of a differential manifold and a Lie algebra acting on the differential functions, to all  $C^\infty$  functions. [3].

It would be useful to introduce a *quantization map*:

$$\mathcal{A} \ni a \mapsto \underline{a} \in \mathcal{A}_\lambda,$$

so that the Eq.3 could be rewritten as:

$$(4) \quad \underline{a} * \underline{b} = \underline{ab} \lambda^{n_1^a n_2^b},$$

It will be useful to rewrite the above quantization form in a more abstract form, using the action of the generators  $p_i$ .<sup>1</sup>

*Remark 4.* The quantization map (3) could be written as:  $\mathcal{A}_\Psi$ :

$$(5) \quad \underline{m}(\underline{a} \otimes \underline{b}) = \underline{m(\Psi^{-1} \triangleright (a \otimes b))},$$

where  $m$  is the multiplication map

$$m : \mathcal{A} \otimes \mathcal{A} \ni a \otimes b \rightarrow ab \in \mathcal{A}.$$

and  $\underline{m}$  is, similarly, the multiplication in deformed algebra, whereas  $\Psi$  is:

$$(6) \quad \Psi_c = \lambda^{-p_1 \otimes p_2},$$

### 3. RIEFFEL'S DEFORMATION BY THE ACTION OF $\mathbb{R}^n$

**3.1. General construction.** Suppose we have an algebra  $A$  and the action of  $V = \mathbb{R}^n$  on this algebra. and a linear map,  $J$ , from  $V'$ , the dual of  $V$  to  $V$ , such that it is skew-symmetric  $J^T = -J$ . Again, one takes a subalgebra of elements, which are  $C^\infty$  vectors in  $A$  for the action of  $\mathbb{R}^n$ .

To make a direct correspondence with the above case of isospectral deformations we restrict ourselves to  $n = 2$ . In analogy with Poisson brackets on the function on a manifold we might define a Poisson bracket:

$$(7) \quad P(a, b) = \sum_i \alpha_{Jr_i}(a) \alpha_{p_i}(b),$$

where  $p_i$  is the basis of  $\mathbb{R}^2$  and  $r_i$  of its dual. Clearly this is independent of the choice of the basis (see [8, 9]) and makes the algebra  $A^\infty$  a strict Poisson \*-algebra.

Then using the oscillatory integrals one can define a deformed product:

$$(8) \quad a \times_J b = \int_V \int_{V'} \alpha_{Jy}(a) \alpha_x(b) e^{2\pi i(y \cdot x)},$$

which could be recognized as a deformation quantization in the direction of the Poisson structure as defined in (7).

---

<sup>1</sup>We use here a particular form of a more general formula derived in [12]

**3.2. Equivalence with isospectral deformations.** To obtain the deformation as defined in (3) we have to modify the expression (8) by allowing arbitrary (not necessarily antisymmetric) operator  $J$ .

Let us calculate explicitly for homogeneous elements  $a$  and  $b$ . We parameterize  $V$  (coordinates  $x$ , basis  $p_i$ ) and  $V'$  (coordinates  $y$ , basis  $e_i$ ), and with a particular choice of the map  $J$ .

$$Je_1 = 0, \quad Je_2 = \theta p_1.$$

Then:

$$a \times_J b = \int_V \int_{V'} d^2x d^2y e^{2\pi i(\theta y_2 n_1^a)} a e^{2\pi i(x_1 n_1^b + x_2 n_2^b)} b e^{2\pi i(y_1 x_1 + y_2 x_2)} = \dots$$

Calculating further using the standard properties of oscillatory integrals we obtain:

$$\begin{aligned} \dots &= ab \int_V d^2x \int_{V'} d^2y e^{2\pi i(\theta n_1^a y_2 + n_2^b x_2 + y_2 x_2)} = \\ &= ab e^{2\pi i \theta n_1^a n_2^b}, \end{aligned}$$

which agrees with the definition for the right twist as defined in (3) with  $\lambda = e^{2\pi i \theta}$ .

Having shown the relation one might use the results valid for the deformation quantization to this particular case, for instance to the noncommutative four-spheres. On the other hand, using similar arguments as in [3] one will obtain a vast family of noncommutative spectral triples and their symmetries, by constructing the deformation quantization as described in [8] of, for instance, spin manifolds.

**Acknowledgements:** While finishing this paper we have learned of a similar work by Joseph Varilly [13], more general picture shall be also presented in a forthcoming work [4].

The author would also like to thank Michel Dubois-Violette, J.M. Gracia-Bondía, Gianni Landi, John Madore, Mario Pachke and Harold Steinacker for helpful discussions on many topics.

## REFERENCES

- [1] A. Connes, *Noncommutative geometry*, Academic Press 1994,
- [2] A. Connes, *Noncommutative geometry Year 2000*, arXiv:math.QA/0011193,
- [3] A. Connes, G. Landi *Noncommutative manifolds, the instanton algebra and isospectral deformations*, arXiv:math.QA/0011194,
- [4] A. Connes, M. Dubois-Violette, in preparation,
- [5] P. P. Kulish, A. I. Mudrov, *Twist-like geometries on a quantum Minkowski space.*, On the occasion of the 65-th birthday of Academician Lyudvig Dmitrievich Faddeev, Tr. Mat. Inst. Steklova 226 (1999), Mat. Fiz. Probl. Kvantovoi Teor. Polya, 97–111

- [6] M.Paschke, *Über nichtkommutative Geometrien, ihre Symmetrien und etwas Hochenergiephysik*, Ph.D. thesis, Mainz 2001, to appear,
- [7] M.Paschke, A.Sitarz, *The geometry of noncommutative symmetries*, Acta Phys.Pol. B31, No 11, (2000)
- [8] M.Rieffel, *Deformation quantization for actions of  $R^d$* , Memoirs AMS, vol 506, AMS, Providence, (1993),
- [9] M.Rieffel, *Quantization and  $C^*$  algebras*, in:  $C^*$  algebras, 1943-1993, Contemporary Mathematics 167, Editor: R.E.Dolan, AMS, Providence, (1994)
- [10] M.Rieffel, *Non-compact quantum groups associated with Abelian subgroups*, Comm.Math.Phys. 171, p 181, (1995)
- [11] M.Rieffel, *Deformation quantization for actions of  $R^d$* , Memoirs AMS, vol 506, AMS, Providence, (1993),
- [12] A.Sitarz, *Twists and spectral triples for isospectral deformations*, to appear,
- [13] J.C.Várilly, *Quantum symmetry groups of noncommutative spheres*, arXiv:math.QA/0102065,

*E-mail address:* Andrzej.Sitarz@th.u-psud.fr

LABORATOIRE DE PHYSIQUE THEORIQUE, UNIVERSITÉ PARIS-SUD, BAT. 210,  
91405 ORSAY CEDEX, FRANCE